ITEP–TH–41/00
hep-th/0007247

On noncommutative vacua and noncommutative solitons

A.S.Gorsky, Y.M.Makeenko and K.G.Selivanov

ITEP, B.Cheremushkinskaya 25, 117259 Moscow

Abstract

We consider noncommutative theory of a compact scalar field. The recently discovered projector solitons are interpreted as classical vacua in the model considered. Localized solutions to the projector equation are pointed out and their brane interpretation is discussed. An example of the noncommutative soliton interpolating between such vacua is given. No strong noncommutativity limit is assumed.

1. Noncommutative theories have been attracted recently a great deal of attention both at perturbative and nonperturbative levels. Even the simplest noncommutative scalar theory reveals a new peculiar object which is absent in commutative theory namely nontrivial solution to the equation of motion at large noncommutativity. It was identified as the noncommutative soliton in the scalar theory with generic potential [1, 2]. Later this solution got interpreted as a brane localized within the brane of higher dimension with the large B field included [3]. This solution finds a proper place in the context of the tachyon condensation [4] as well as in the open string field theory formalism [5]. Recently new solutions implying the restoration of the full gauge symmetry outside the soliton core have been found [6].

In this note we would like to consider the theory with a compact scalar field. In this case we shall show that noncommutativity amounts to rather rich vacuum structure. Interestingly the projector solitons discussed above,

despite the fact that they have a nontrivial dependence on noncommutative coordinates, play in our case the role of classical vacua additional to the simple "commutative" ones. A domain wall type soliton interpolating between the vacua is found. We also construct a new localized solution in the case of large noncommutativity. The brane interpretation of the solutions considered is given.

2. Let us remind the description of the noncommutative solitons in terms of the projector operators. We shall work in (2+1)D space with Euclidean signature, two coordinates being noncommutative:

$$[x, y] = -i. \tag{1}$$

In this equation we assume that the noncommutative coordinates have been rescaled to get rid of the noncommutativity parameter θ from the right hand side; it will then be present in front of a kinetic term (cf. [1]). In [1] solitons in a theory of noncommutative scalar field were obtained in the limit of strong noncommutativity ($\theta \to \infty$). In that limit one can neglect kinetic term and the field equation reduces to

$$\frac{\partial}{\partial \phi} V(\phi) = 0, \tag{2}$$

where $V(\phi)$ is a potential, $V(\phi) = 1/2 m^2 \phi^2 + \sum_n 1/n! \lambda_n \phi^n$. The solutions to Eq.(2) were found using the ansatz

$$\phi(x, y) = \sum_n \alpha_n P_n(x, y) \tag{3}$$

where $P_n(x, y)$ is a set of projectors,

$$P_n(x, y) * P_m(x, y) = \delta_{n,m} P_n(x, y), \tag{4}$$

and the star-product $f(x^\mu) * g(x^\nu) = e^{-i/2 \epsilon_{\mu\nu} \frac{\partial}{\partial \eta_\mu} \frac{\partial}{\partial \rho_\nu}} f(\eta_\mu) g(\rho_\nu)|_{\eta_\mu = x^\mu, \rho_\nu = x^\nu}$ have been used.

In view of Eq.(4) the field equation (2) further reduces to

$$\frac{\partial}{\partial \phi} V(\alpha_n) = 0 \tag{5}$$

so the parameters α_n must be extrema of the potential. This way the problem of constructing solutions has been reduced to the problem of constructing projectors.

Radially symmetric projectors, $P_n(r^2)$, are most conveniently described (see [2]) in terms of a generating function $P(r^2, u)$,

$$P(r^2, u) = \sum_n u^n P_n(r^2). \tag{6}$$

Eq.(4) is then rewritten as

$$P(r^2, u) * P(r^2, v) = P(r^2, uv) \tag{7}$$

and the completeness condition

$$\sum_n P_n = 1 \tag{8}$$

is rewritten as

$$P(r^2, 1) = 1. \tag{9}$$

The solution to Eqs.(7), (9) reads

$$P(r^2, u) = \frac{2}{u+1} e^{\frac{u-1}{u+1} r^2} \tag{10}$$

which is a generating functions for the Laguerre polynomials. The construction of projectors completes the construction of the noncommutative solitons [1].

Our point here is to interpret those solutions rather as classical vacuum states, while the proper solitons are solutions interpolating between these vacua. To illustrate this point, let us consider the following noncommutative field theory with the compact scalar field:

$$L = \int \frac{1}{2\theta} \partial_A g^{-1} \partial_A g + 2 - g - g^{-1} \tag{11}$$

where $g = e_*^{i\phi} = 1 + i\phi + 1/2 i\phi * i\phi + \ldots$ and g^{-1} is inverse to g in the sense of the star-product:

$$g^{-1} * g = g * g^{-1} = 1. \tag{12}$$

The derivatives ∂_A include derivatives with respect to the noncommutative coordinates ∂_μ and derivative with respect to (Euclidean or Minkowski) time τ.

Notice that the kinetic term in Eq.(11) can be considered as the one of a noncommutative $U(1)$ sigma-model. In the commutative limit it reduces to the common kinetic term for a scalar field so it is a legal noncommutative generalization of the latter. The potential term reduces in the commutative limit to $2(1 - \cos\phi)$. In principle one can consider additional vacua in the theory without the potential term but it will be used to construct time-dependent solutions interpolating between different vacua. By the way, in 2D one can add a Wess-Zumino term to the Lagrangian (11) obtaining this way a noncommutative integrable sin-Gordon model. At rational values of the noncommutativity parameter the noncommutative sin-Gordon model on a torus is Morita equivalent to a nonabelian sin-Gordon model, a member of the family of Polyakov's nonabelian Toda theories. The noncommutative sin-Gordon model was discussed in the context of the noncommutative bosonization in [7].

Let us return back to the model Eq.(11). Corresponding field equation reads

$$\frac{1}{\theta}\partial_A\left(g^{-1} * \partial_A g\right) = g - g^{-1}. \tag{13}$$

In complete analogy with the commutative case there are classical vacua solutions

$$\phi_n = 2\pi n. \tag{14}$$

In addition there are solutions of the type of Eq.(3) with

$$\alpha_n = 2\pi m_n \tag{15}$$

where m_n is a set of integers (if one considers a theory without the potential, α_n are no longer restricted to be integer). These are obviously solutions of Eq.(13) in view of the fact that when ϕ is of the form of Eq.(3),

$$g = e_*^{i\phi} = \sum_n e^{i\alpha_n} P_n \tag{16}$$

where, we remind, $e_*^{i\phi}$ is the star-exponential of $i\phi$ and the exponential of α on the right hand side is the usual exponential. When α_n are as in Eq.(15), g is equal to 1, so such states are most natural to interpret as the additional

classical vacua. We would like to emphasize that unlike [1] we do not restrict ourselves to the strong noncommutativity limit and vacuum states exist at arbitrary θ.

Let us discuss in more detail the vacuum state of the type

$$\begin{aligned} \alpha_n &= \alpha_+,\ n-\mathrm{even}\,, \\ \alpha_n &= \alpha_-,\ n-\mathrm{odd}\,. \end{aligned} \tag{17}$$

Then

$$\phi = \alpha_+ \frac{1+P(r^2,-1)}{2} + \alpha_- \frac{1-P(r^2,-1)}{2}. \tag{18}$$

Using the fact that

$$P(r^2,-1) = \pi\delta^2(x) \tag{19}$$

(notice that, in view of Eq.(7), $\pi\delta^2(x) * \pi\delta^2(x) = 1$) one can also rewrite the field as

$$\phi = \frac{\alpha_+ + \alpha_-}{2} + \frac{\alpha_+ - \alpha_-}{2}\pi\delta^2(x). \tag{20}$$

According to the general discussion above, it is a classical vacuum state, providing $\alpha_{+/-} = 2\pi m_{+/-}$. A few additional remarks concerning this solution are in order. First, the soliton can be placed at an arbitrary point so a nontrivial moduli space of the solutions appears. We shall describe it in a moment. One could ask if a superposition of the localized solitons placed at different positions in the noncommutative plane is again a solution to the equation of motion. The answer is "no" due to the following identity

$$\pi\delta^2(x-a) * \pi\delta^2(x-b) = e^{i\Phi(a,b,x)} \tag{21}$$

where $\Phi(a,b,x) = 2\epsilon_{\mu\nu}(a_\mu b_\nu + b_\mu x_\nu + x_\mu a_\nu)$ is the flux of the B field through the triangle with the vertices at a, b and x.

The solution (3) can be transformed into another solution by a star-unitary transformation

$$\phi' = \omega * \phi * \omega^{-1}. \tag{22}$$

This is again a solution since $\omega * \omega^{-1} = 1$. The translation discussed above is the simplest example of such a transformation given by $\omega = e^{ik_x x + ik_y y}$. Another example is given by the Gaussian function

$$\omega(x,y) = \sqrt{1+\alpha\beta-\gamma^2}\; e^{i\alpha x^2 + i\beta y^2 + 2i\gamma xy} \tag{23}$$

which is star-unitary. The transformation generated by this function keeps the functional form of $\phi(x,y)$ unchanged but rotates x,y in the noncommutative plane by an $sl(2;R)$ matrix into

$$\begin{aligned} x' &= \frac{(1+\gamma^2)^2-\alpha\beta}{1+\alpha\beta-\gamma^2}\,x+\frac{2\beta}{1+\alpha\beta-\gamma^2}\,y\,, \\ y' &= -\frac{2\alpha}{1+\alpha\beta-\gamma^2}\,x+\frac{(1-\gamma^2)^2-\alpha\beta}{1+\alpha\beta-\gamma^2}\,y\,. \end{aligned} \tag{24}$$

In general, the new solution is no longer spherically symmetric. Some of the star-unitary transformation leave the solution (3) invariant. An example of such a star-unitary function is

$$\omega(x,y)=\pi\delta(x)\delta(y) \tag{25}$$

which only inverts the signs of x and y.

3. Thus the structure of vacua in the noncommutative framework is much more involved than in the commutative one. Let us now discuss solitons of the type of domain walls, i.e., solutions interpolating between different vacua of the type of Eq.(3). We shall now assume that coefficients α in Eq.(3) are τ-dependent but independent of the noncommutative coordinates. In this assumption the field equation (13) reduces to

$$i/\theta\sum_n(\frac{\partial}{\partial\tau})^2\alpha_nP_n+1/\theta\sum_{n,m}e^{i(\alpha_m-\alpha_n)}\partial_\mu\left(P_n*\partial_\mu P_m\right)=\sum_n(e^{i\alpha_n}-e^{-i\alpha_n})P_n \tag{26}$$

The term with derivatives of the projectors, $\partial_\mu\left(P_n*\partial_\mu P_m\right)$, can be computed in terms of the projectors using the generating function:

$$\begin{aligned} \Big(\partial_\mu P(r^2,u)\Big)*\Big(\partial_\mu P(r^2,v)\Big) &= (u-1)(v-1)\partial_{uv}\Big((uv+1)P(r^2,uv)\Big) \\ \partial_\mu\Big(P(r^2,u)*\partial_\mu P(r^2,v)\Big) &= (u+1)(v-1)\partial_{uv}\Big((uv-1)P(r^2,uv)\Big) \end{aligned} \tag{27}$$

We shall however not need it because we make a further assumption under which that term disappears. Namely, we assume that all α's differ from each other by only integers, that is,

$$\alpha_n=\bar{\alpha}+2\pi m_n. \tag{28}$$

Then Eq.(26) reduces to an ordinary differential equation for $\bar{\alpha}$,

$$(\frac{\partial}{\partial\tau})^2\bar{\alpha} = 2\theta\sin(\bar{\alpha}), \tag{29}$$

the solitonic solution to which is well known to be

$$\bar{\alpha} = \frac{4}{\sqrt{2\theta}}\arctan(e^{\sqrt{2\theta}\tau}). \tag{30}$$

For this solution all α_n's change by 2π when τ runs from $-\infty$ to ∞.

One can introduce a topological charge associated to solitons, a generalization of $\int_C d\phi$ to the noncommutative framework:

$$Q = \frac{1}{2\pi i}\int d^2x \int_C g^{-1} * dg \tag{31}$$

where the first integration is over the noncommutative plane (a sort of trace) and the contour C is assumed to begin and end at the same point of the noncommutative plane. Since the integrated form is a flat connection, Q does not depend on the shape of C and does not change under the star-unitary transformation (22). When g is of the form Eq.(16) with possibly τ-dependent α's, one obtains

$$Q = \frac{1}{2\pi}\int d^2x \sum_n \alpha_n|_-^+ P_n = \sum_n \alpha_n|_-^+ \tag{32}$$

where $\alpha_n|_-^+$ is a total change of α_n along the path C. The projectors are easily integrated using the generating function Eq.(10). One can also use non-integrated version of topological charge which depends on a point on the noncommutative plane.

4. Let us discuss the brane interpretation of the solutions discussed above. First let us consider the new solution to the projector equation we have constructed. It was shown in [3] that noncommutative soliton at large noncommutativity is nothing but a brane of lower dimension inside the brane if the tachyon Lagrangian is considered. For instance the Gaussian projector solution for bosonic string was interpreted as D23 brane within D25 brane. The key point of this identification is the reproducing of the correct tension of D23 brane from the action calculated on the noncommutative soliton. However this interpretation is a little bit subtle since the size of the Gaussian soliton

is nonzero. For the localized delta function solution which is perfectly localized this subtlety is resolved and to confirm the identification of our localized solution as a D brane we have to calculate its tension.

For example the action on the localized soliton in bosonic string looks as

$$S = -\frac{g_s T_{25}}{G_s} \int d^{24} \int d^2 \sqrt{G} \phi(r) \tag{33}$$

where $\phi(r)$ is the localized soliton, g_s is a string coupling, G is a metric and $G_s = \frac{g_s \sqrt{G}}{2\pi B \sqrt{g} \alpha'}$. The direct calculation amounts to the correct tension of the brane. Let us also note that in the type II theory where the tachyon potential is definitely even one can construct the additional solution based on the relation $\Delta^3 = \Delta$ instead of the projector one solving the equation of motion. This solution

$$\Delta(x) = \pi \delta^2(x) \tag{34}$$

admits the similar brane interpretation as well.

Let us turn to the role of the solution in the theory of the compact scalar which it presumably plays in the brane context. To this aim remind that the complex tachyon field is certainly present for the brane-antibrane system [4] and its potential has a form of the “Mexican hat”. Then at the vacuum manifold we have $T * \bar{T} = const$ therefore the vacuum valley is parameterized by the phase of the complex scalar which is definitely compact. Therefore we arrive at the theory with the action we have discussed before. Hence we can expect nontrivial vacuum states involving projectors in the theory of the complex tachyon. It is known that the BPS D(p-2) brane arises as vortex in the complex tachyon field in the $Dp - \bar{D}p$ system [4] (noncommutative vortexes were also discussed [8]) Such stable configuration can be also derived for the additional noncommutative vacuum state where the tachyon phase has nontrivial space dependence. We shall discuss this point in more details elsewhere.

5. In this letter we have described a rich vacuum structure in the noncommutative theory of the compact scalar field. The example of the kink type solution interpolating between two vacua of the theory is presented. New solutions to the projector equations are found and their brane interpretation is given. Let us also note that the noncommutative solitons where recently discussed in the context of Quantum Hall effect [9]. It seems that the new

localized noncommutative solitons we have found are better candidates for the skyrmion spin textures then Gaussian solitons discussed in that paper.

This work was supported in part by CRDF grant RP1-2108. The work of A.G. is supported in part by grant INTAS-99-1705 and K.S. by INTAS-97-0103. A.G. thanks J. Ambjørn for the hospitality at NBI where the part of the work has been done.

References

[1] R. Gopakumar, S. Minwalla and A. Strominger, hep-th/0003160

[2] K. Dasgupta, S. Mukhi and G. Rajesh, hep-th/0005006

[3] J. Harvey, P. Kraus, F. Larsen and E. Martinec, hep-th/0005031

[4] A. Sen, hep-th/9805170, hep-th/9808041, hep-th/9902105

[5] E. Witten, hep-th/0006071

[6] R. Gopakumar, S. Minwalla and A. Strominger, hep-th/0007226

[7] C. Nunez, K. Olsen and R. Schiappa, hep-th/0005059

[8] D. Jatkar, G. Mandal and S. Wadia , hep-th/0007078
A. Polychronakos, hep-th/0007043

[9] V. Pasquier, hep-th/0007176